A Note on Optimal Stopping in Models with Delay

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We consider an optimal stopping problem in a certain model described by a stochastic delay differential equation. We reduce the initial problem to a free-boundary problem of parabolic type and prove the corresponding verification assertion. We also give an example of such an optimal stopping problem related to mathematical finance.

1 Introduction

It is known that optimal stopping problems form an important class of optimal control problems having applications in stochastic calculus (maximal inequalities), statistics (sequential analysis) and mathematical finance (American options). The results about the relationship between optimal stopping problems for Markov processes and free-boundary problems for partial differential equations often give an opportunity to obtain explicit solutions in some particular cases (see e.g. Shiryaev [11] or [12]). In the present paper we consider some optimal stopping problems in a model with processes solving stochastic delay differential equations.

In recent years several stochastic control problems for models described by stochastic delay differential equations were studied. Øksendal and Sulem [7] proved maximum principles for certain classes of such models and applied

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them to solving some problems related to finance. Elsanosi, Øksendal and Sulem [2] proved a verification theorem of variational inequality type and applied it to finding explicit solutions for some classes of optimal harvesting delay problems. Larssen and Risebro [4] proved that under certain conditions the value function of such optimal control problems turns out to be the unique viscosity solution of the Hamilton-Jacobi-Bellman equation. The aim of this paper is to show the way of solving optimal stopping problems in some specific models described by stochastic delay differential equations.

The paper is organized as follows. In Section 2 we formulate an optimal stopping problem in a model described by a stochastic delay differential equation with an exponential delay measure on an infinite interval, which is equivalent to a model with two-dimensional Markov process. In Section 3 we derive an associated free-boundary problem of parabolic type and prove the verification assertion showing that the solution of the free-boundary problem turns out to be a solution of the initial optimal stopping problem. In Section 4 we consider some special cases of the problem where the two-dimensional problem is reduced to a one-dimensional one and give an example of such an optimal stopping problem related to mathematical finance.

2 Formulation of the problem

2.1. Suppose that on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ there exists a standard Wiener process $W = (W_t)_{t\geq 0}$ and a continuous process $X = (X_t)_{t\geq 0}$ solving the stochastic differential equation:

$$dX_t = \beta(X_t, Y_t)\, dt + \gamma(X_t, Y_t)\, dW_t, \quad X_0 = x, \tag{2.1}$$

where the process $Y = (Y_t)_{t\geq 0}$ is defined by

$$Y_t = \int_{-\infty}^{0} e^{\lambda s} X_{t+s}\, ds, \quad X_t = X_t^0 \quad \text{for} \quad t \leq 0, \tag{2.2}$$

for some constant $\lambda > 0$ and a deterministic (continuous) function $X^0 = (X_t^0)_{t\leq 0}$. It is further assumed that the functions $\beta(x, y)$ and $\gamma(x, y)$, $(x, y) \in \mathbb{R}^2$, satisfy a Lipschitz condition, that is, there exists a constant $C > 0$ such that

$$[\beta(x, y) - \beta(x', y')]^2 + [\gamma(x, y) - \gamma(x', y')]^2 \leq C[(x - x')^2 + (y - y')^2] \tag{2.3}$$

for all $(x,y),(x',y') \in \mathbb{R}^2$. Our goal is to compute the value

$$V = \sup_{\tau} E\left[e^{-r\tau} h(X_\tau, Y_\tau) - \int_0^\tau e^{-rt} a(X_t, Y_t)\, dt\right] \tag{2.4}$$

where the supremum is taken over all stopping times τ with respect to the filtration $(\mathcal{F}_t^X)_{t\geq 0}$ generated by the process X, i.e. $\mathcal{F}_t^X = \sigma\{X_s \,|\, 0 \leq s \leq t\}$ for $t \geq 0$, such that

$$E\left[\int_0^\tau e^{-rt} a(X_t, Y_t)\, dt\right] < \infty \tag{2.5}$$

and to determine an optimal stopping time τ_* at which the supremum in (2.4) is attained. Here $r \geq 0$ is a constant discount rate, $a(x,y) \geq 0$ is a continuous bounded cost function, and $h(x,y) \geq 0$ is a continuous bounded gain function for all $(x,y) \in \mathbb{R}^2$.

2.2. Let us introduce the process $Z = (Z_t)_{t\geq 0}$ defined by:

$$Z_t = X_t - \lambda Y_t \tag{2.6}$$

which, by means of Itô's formula (see e.g. [5; Theorem 4.4] or [3; Chapter I, Theorem 4.57]), solves the stochastic differential equation:

$$dZ_t = \mu(Z_t, Y_t)\, dt + \sigma(Z_t, Y_t)\, dW_t, \quad Z_0 = z, \tag{2.7}$$

with $\mu(z,y) = \beta(z + \lambda y, y) - \lambda z$ and $\sigma(z,y) = \gamma(z + \lambda y, y)$ for $(z,y) \in \mathbb{R}^2$. Then equation (2.2) and simple calculations yield that the process $Y = (Y_t)_{t\geq 0}$ admits the representation:

$$dY_t = Z_t\, dt, \quad Y_0 = y, \tag{2.8}$$

for some $(z,y) \in \mathbb{R}^2$. Note that from (2.2) and (2.6) it follows that $Z_0 = z$ in (2.7) and $Y_0 = y$ in (2.8) can be straightforwardly expressed by means of X_t^0, $t \leq 0$.

Observe that since the functions $\beta(x,y)$ and $\gamma(x,y)$, $(x,y) \in \mathbb{R}^2$, are assumed to satisfy the Lipschitz condition (2.3), it easily follows that the functions $\mu(z,y)$ and $\sigma(z,y)$, $(z,y) \in \mathbb{R}^2$, also satisfy a Lipschitz condition. Hence, by remark to [5; Theorem 4.6] or [6; Theorem 5.2.1] we may conclude that the process $(Z_t, Y_t)_{t\geq 0}$ is the unique strong solution of the (two-dimensional) stochastic differential equation (2.7) - (2.8), and thus, by virtue of [3; Chapter III, Theorem 2.34] or [6; Theorem 7.2.4], it is a (strong) Markov process with respect to its natural filtration, which due to the one-to-one correspondence (2.6) between the processes $(X_t, Y_t)_{t\geq 0}$ and $(Z_t, Y_t)_{t\geq 0}$ coincides with $(\mathcal{F}_t^X)_{t\geq 0}$. From now on we assume that the state space of the process $(Z_t, Y_t)_{t\geq 0}$ is $\mathbb{R}^2$.

2.3. We can thus reduce the stopping problem (2.4) for the process with time delay to the following optimal stopping problem for a two-dimensional diffusion process:

$$V(z,y) = \sup_{\tau} E_{z,y}\left[e^{-r\tau} g(Z_\tau, Y_\tau) - \int_0^\tau e^{-rt} c(Z_t, Y_t)\, dt\right] \tag{2.9}$$

where $P_{z,y}$ denotes the law of the diffusion process $(Z_t, Y_t)_{t\geq 0}$ starting at the point (z,y) and solving equation (2.7) - (2.8), τ denotes any stopping time of $(Z_t, Y_t)_{t\geq 0}$ such that

$$E\left[\int_0^\tau e^{-rt} a(X_t, Y_t)\, dt\right] < \infty \tag{2.10}$$

and where we have set $c(z,y) = a(z + \lambda y, y)$, $g(z,y) = h(z + \lambda y, y)$ for all $(z,y) \in \mathbb{R}^2$.

3 The free-boundary problem

3.1. It follows from the general optimal stopping theory (cf. [11; Chapter III]) that the continuation region C and the stopping region D for the optimal stopping problem (2.9) are given by:

$$C = \{(z,y) \in \mathbb{R}^2 \,|\, V(z,y) > g(z,y)\} \tag{3.1}$$

and

$$D = \{(z,y) \in \mathbb{R}^2 \,|\, V(z,y) = g(z,y)\} \tag{3.2}$$

respectively. We will further assume that the functions c and g satisfy some regularity conditions which imply the existence of a function of bounded variation b such that the stopping time:

$$\tau_* = \inf\{t \geq 0 \,|\, Z_t \leq b(Y_t)\} \tag{3.3}$$

turns out to be optimal in (2.9) whenever it satisfies (2.10), so that the regions (3.1) and (3.2) take the form:

$$C = \{(z, y) \in \mathbb{R}^2 \mid z > b(y)\} \tag{3.4}$$

and

$$D = \{(z, y) \in \mathbb{R}^2 \mid z \le b(y)\} \tag{3.5}$$

respectively.

Standard arguments based on the application of Itô's formula imply that the infinitesimal operator $\mathbb{L}$ of the process $(Z_t, Y_t)_{t \ge 0}$ acts on a function $f \in C^{2,1}(\mathbb{R}^2)$ like:

$$(\mathbb{L}f)(z, y) = \left(\mu(z, y) \frac{\partial f}{\partial z} + \frac{\sigma^2(z, y)}{2} \frac{\partial^2 f}{\partial z^2} + z \frac{\partial f}{\partial y} \right) (z, y) \tag{3.6}$$

for all $(z, y) \in \mathbb{R}^2$.

Assuming that g is $C^{2,1}$ in $\mathbb{R}^2$ and based on the results about the relationship between optimal stopping problems and free-boundary problems for partial differential equations (see e.g. [11; Chapter III, Section 8]), in order to find analytic expressions for the value function V from (2.9) and the boundary b from (3.3) we may formulate the following free-boundary problem for the parabolic type operator $\mathbb{L}$ acting like in (3.6):

$$[-rV + (\mathbb{L}V)](z, y) = c(z, y) \quad \text{for} \quad z > b(y) \tag{3.7}$$

$$V(z, y)\big|_{z=b(y)+} = g(b(y), y) \tag{3.8}$$

$$\frac{\partial V}{\partial z}(z, y)\big|_{z=b(y)+} = \frac{\partial g}{\partial z}(z, y)\big|_{z=b(y)-} \tag{3.9}$$

$$V(z, y) > g(z, y) \quad \text{for} \quad z > b(y) \tag{3.10}$$

$$V(z, y) = g(z, y) \quad \text{for} \quad z < b(y) \tag{3.11}$$

$$[-rV + (\mathbb{L}V)](z, y) < c(z, y) \quad \text{for} \quad z < b(y) \tag{3.12}$$

where the *instantaneous stopping* equation (3.8) and the *smooth fit* equation (3.9) are satisfied for all $y \in \mathbb{R}$. It follows by the superharmonic characterization of the value function (see [1] or [11; Chapter III]) that V from (2.9) is the smallest function satisfying (3.7) - (3.8) and (3.10) - (3.12).

3.2. Let us verify that system (3.7) - (3.12) implies a solution of the optimal stopping problem (2.9). For the proof of the following verification assertion we use the same stucture as in the proofs of the corresponding assertions from [11; Chapter IV], [6; Chapter X], [9] and [8].

Theorem 3.1. *Suppose that the function V_* and the boundary b is the unique solution of the free-boundary problem (3.7) - (3.12). Then V_* coincides with the value function V from (2.9) and the stopping time τ_* from (3.3) is optimal in (2.9) whenever is satisfies (2.10).*

Proof. Using the fact that the smooth fit condition (3.9) implies that $z \mapsto V_*(z,y)$ is C^1 at $b(y)$ for all $y \in \mathbb{R}$, we obtain by Itô's formula for any $(z,y) \in \mathbb{R}^2$:

$$e^{-rt} V_*(Z_t, Y_t) = V_*(z,y) + \int_0^t e^{-rs} [-rV_* + (\mathbb{L}V_*)](Z_s, Y_s)\, ds + M_t \tag{3.13}$$

where $M = (M_t)_{t \ge 0}$ defined by

$$M_t = \int_0^t e^{-rs} \sigma(Z_s, Y_s) \frac{\partial V_*}{\partial z}(Z_s, Y_s)\, dW_s \tag{3.14}$$

is a continuous local martingale.

Since (3.7) and (3.12) imply that $[-rV_* + (\mathbb{L}V_*)](z,y) \le c(z,y)$ for all $(z,y) \in \mathbb{R}^2$, using the fact that the values $(\mathbb{L}V_*)(z,y)$ for $z = b(y)$ can be set arbitrarily, from (3.13) we get:

$$e^{-r\tau} V_*(Z_\tau, Y_\tau) - \int_0^\tau e^{-rs} c(Z_s, Y_s)\, ds \le V_*(z,y) + M_\tau \tag{3.15}$$

for any stopping time τ of the process $(Z_t, Y_t)_{t \ge 0}$. Moreover, using that (3.10) - (3.11) yield $V_*(z,y) \ge g(z,y)$ for all $(z,y) \in \mathbb{R}^2$, from (3.15) it follows that:

$$e^{-r\tau} g(Z_\tau, Y_\tau) - \int_0^\tau e^{-rs} c(Z_s, Y_s)\, ds \le V_*(z,y) + M_\tau \tag{3.16}$$

for any stopping time τ of $(Z_t, Y_t)_{t \ge 0}$.

Let us choose a sequence $(\sigma_n)_{n \in \mathbb{N}}$ of bounded stopping times for $M = (M_t)_{t \ge 0}$. Then taking in (3.16) expectation with respect to the measure $P_{z,y}$, by means of the optional sampling theorem (see e.g. [3; Chapter I, Theorem 1.39] or [10; Chapter II, Theorem 3.2]) we get:

$$E_{z,y} \left[e^{-r(\tau \wedge \sigma_n)} g(Z_{\tau \wedge \sigma_n}, Y_{\tau \wedge \sigma_n}) - \int_0^{\tau \wedge \sigma_n} e^{-rs} c(Z_s, Y_s)\, ds \right] \tag{3.17}$$
$$\le V_*(z,y) + E_{z,y}[M_{\tau \wedge \sigma_n}] = V_*(z,y).$$

Thus, letting $n \to \infty$ and using Fatou's lemma, we obtain:

$$E_{z,y}\left[e^{-r\tau}g(Z_\tau,Y_\tau)-\int_0^\tau e^{-rs}c(Z_s,Y_s)\,ds\right]\le V_*(z,y) \tag{3.18}$$

for any $(z,y)\in\mathbb{R}^2$.

On the other hand, in order to show that the equality in (3.18) is attained at τ_* from (3.3) we observe that from (3.7), (3.8), (3.11), and (3.13) with t replaced by τ_* it follows that:

$$e^{-r\tau_*}g(Z_{\tau_*},Y_{\tau_*})-\int_0^{\tau_*} e^{-rs}c(Z_s,Y_s)\,ds=V_*(z,y)+M_{\tau_*}. \tag{3.19}$$

Thus, the assumption that g is bounded implies that the process $(M_{\tau_*\wedge t})_{t\ge0}$ turns out to be an $(\mathcal{F}_t^X,P_{z,y})$-uniformly integrable martingale whenever τ_* satisfies (2.10). Therefore, taking in (3.19) expectation with respect to the measure $P_{z,y}$, we obtain:

$$E_{z,y}\left[e^{-r\tau_*}g(Z_{\tau_*},Y_{\tau_*})-\int_0^{\tau_*} e^{-rs}c(Z_s,Y_s)\,ds\right]=V_*(z,y), \tag{3.20}$$

which yields the desired assertion. □

Remark 3.2. In particular cases the system (3.7) - (3.12) can be solved by numerical methods.

4 Some special cases

4.1. Let us now assume that in (2.1) we have $\beta(x,y)=\eta(x-\lambda y)$ and $\gamma(x,y)=\theta(x-\lambda y)$, so that the process $X=(X_t)_{t\ge0}$ solves the stochastic differential equation:

$$dX_t=\eta(X_t-\lambda Y_t)\,dt+\theta(X_t-\lambda Y_t)\,dW_t,\quad X_0=x, \tag{4.1}$$

where the process $Y=(Y_t)_{t\ge0}$ is defined in (2.2). In this case from the arguments of Subsection 2.2 it follows that the process $Z=(Z_t)_{t\ge0}$ defined in (2.6) solves the stochastic differential equation:

$$dZ_t=[\eta(Z_t)-\lambda Z_t]\,dt+\theta(Z_t)\,dW_t,\quad Z_0=z, \tag{4.2}$$

and thus, turns out to be an $(\mathcal{F}_t^X,P)$-(strong) Markov process.

Let us further assume that $c(z,y) = p(z)e^{\nu y}$ and $g(z,y) = q(z)e^{\nu y}$ for some functions $p(z)$ and $q(z)$ and a constant $\nu \in \mathbb{R}$. In this case by means of easy calculations from (2.8) we infer that the value function (2.9) takes the form $V(z,y) = e^{\nu y}U(z)$ with:

$$U(z) = \sup_{\tau} E_z \left[e^{-\int_0^\tau (r-\nu Z_t)\,dt} q(Z_\tau) - \int_0^\tau e^{-\int_0^t (r-\nu Z_s)\,ds} p(Z_t)\,dt \right] \tag{4.3}$$

where P_z denotes the law of the diffusion process $Z = (Z_t)_{t\geq 0}$ starting at the point z and solving equation (4.2) for all $z \in \mathbb{R}$ and τ denotes any stopping time of Z. In the case when $\nu > 0$ we will assume that the state space of the process Z is $(-\infty, r/\nu)$.

Based on the same arguments as in Subsection 3.1 we may formulate the following (one-dimensional) free-boundary problem for the value function U from (4.3) and the boundary b from (3.3) which turns out to be constant in this case:

$$-(r-\nu z)U(z) + (\eta(z) - \nu z)U'(z) + \frac{\theta^2(z)}{2}U''(z) = p(z), \quad z > b, \tag{4.4}$$

$$U(z)\big|_{z=b+} = q(b), \tag{4.5}$$

$$U'(z)\big|_{z=b+} = q'(z)\big|_{z=b-}, \tag{4.6}$$

$$U(z) > q(z), \quad z > b, \tag{4.7}$$

$$U(z) = q(z), \quad z < b, \tag{4.8}$$

$$-(r-\nu z)U(z) + (\eta(z) - \nu z)U'(z) + \frac{\theta^2(z)}{2}U''(z) < p(z), \quad z < b. \tag{4.9}$$

4.2. Let us finally consider a particular case of the optimal stopping problem (2.4) which is related to mathematical finance. For this we suppose that $X = (X_t)_{t\geq 0}$ solving (4.1) describes the logarithm of the price of some risky asset (e.g. stock) on a financial market so that $Z = (Z_t)_{t\geq 0}$ defined in (2.6) and solving (4.2) expresses the deviation of the logarithm of the present price from its exponentially weighted average. It is seen that if in (4.1) we have $\eta(z) = -\theta^2(z)/2$ for all $z \in \mathbb{R}$, then the initial measure P turns out to be a *martingale measure* for the discounted price process $S = (S_t)_{t\geq 0}$ given by $S_t = e^{-rt+X_t}$ for all $t \geq 0$.

More specifically, let us consider the problem of pricing the following special perpetual average put option of American type:

$$V = \sup_{\tau} E \left[e^{-r\tau} (Ke^{\lambda Y_\tau} - e^{X_\tau})^+ \right]. \tag{4.10}$$

Standard arguments show that the related optimal stopping problem of the type (2.9) is given by:

$$V(z,y) = \sup_{\tau} E_{z,y}\left[e^{-r\tau+\lambda Y_\tau}(K-e^{Z_\tau})^+\right] = e^{\lambda y}U(z) \tag{4.11}$$

with

$$U(z) = \sup_{\tau} E_z\left[e^{-\int_0^\tau (r-\lambda Z_t)\,dt}(K-e^{Z_\tau})^+\right] \tag{4.12}$$

so that in the related free-boundary problem (4.4) - (4.9) we have $p(z)=0$, $q(z)=(K-e^z)^+$ and $\nu=\lambda>0$.

Remark 4.1. The model introduced in this section reflects the empirical evidence that the financial markets are more volatile whenever the actual prices deviate significantly from their past values.

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